

A PROOF OF THE INTEGRAL IDENTITY CONJECTURE, II

LÊ QUY THUONG

ABSTRACT. The present article is a continuation of the work by the same author entitled *Proofs of the integral identity conjecture over algebraically closed fields*, Duke Mathematical Journal **164** (2015), 157–194. Its important advance is that we are able to prove the full conjecture (up to localization) without any additional hypothesis. The essential improvement is well based on a rationality result in Cluckers-Loeser’s motivic integration published as Raf Cluckers and François Loeser, *Constructible motivic functions and motivic integration*, Inventiones Mathematicae **173** (2008), 23–121.

1. INTRODUCTION

Throughout this article, let k be a field of characteristic zero, not necessarily an algebraic closed field. We shall consider the group schemes

$$\mu_m = \mathrm{Spec}k[t]/(t^m - 1), \quad m \in \mathbb{N}_{>0}.$$

These schemes admit obvious morphisms $\mu_{mn} \rightarrow \mu_m$ given by $\xi \mapsto \xi^n$, hence define a projective system, whose limit will be denoted by $\hat{\mu}$. For any k -variety \mathcal{X} , one defines in [3, 4] the Grothendieck rings $K_0(\mathrm{Var}_{\mathcal{X}})$ and $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathcal{X}})$ of the category of \mathcal{X} -varieties and the category of \mathcal{X} -variety endowed with a good $\hat{\mu}$ -action, respectively. Let \mathbb{L} be the class of $\mathcal{X} \times \mathbb{A}_k^1 \rightarrow \mathcal{X}$ in $K_0(\mathrm{Var}_{\mathcal{X}})$ and let

$$\mathcal{M}_{\mathcal{X}} := K_0(\mathrm{Var}_{\mathcal{X}})[\mathbb{L}^{-1}] \quad \text{and} \quad \mathcal{M}_{\mathcal{X}}^{\hat{\mu}} := K_0^{\hat{\mu}}(\mathrm{Var}_{\mathcal{X}})[\mathbb{L}^{-1}].$$

Furthermore, we denote by $\mathcal{M}_{\mathcal{X}, \mathrm{loc}}$ (resp. $\mathcal{M}_{\mathcal{X}, \mathrm{loc}}^{\hat{\mu}}$) the localization of $\mathcal{M}_{\mathcal{X}}$ (resp. $\mathcal{M}_{\mathcal{X}}^{\hat{\mu}}$) with respect to the multiplicative set generated by $1 - \mathbb{L}^i$, $i \in \mathbb{N}_{>0}$. Composing with $\mathcal{X} \rightarrow \mathrm{Spec}k$ yields a morphism of groups

$$\int_{\mathcal{X}} : \mathcal{M}_{\mathcal{X}, \mathrm{loc}}^{\hat{\mu}} \rightarrow \mathcal{M}_{k, \mathrm{loc}}^{\hat{\mu}}.$$

For \mathcal{M} being one of the rings $\mathcal{M}_{\mathcal{X}}$, $\mathcal{M}_{\mathcal{X}, \mathrm{loc}}$, $\mathcal{M}_{\mathcal{X}}^{\hat{\mu}}$ and $\mathcal{M}_{\mathcal{X}, \mathrm{loc}}^{\hat{\mu}}$, let $\mathcal{M}[[T]]$ be the ring of formal series in one variable T with coefficients in \mathcal{M} , and let $\mathcal{M}[[T]]_{\Gamma}$ be the sub- \mathcal{M} -module of $\mathcal{M}[[T]]$ generated by 1 and by finite products of terms $\mathbb{L}^a T^b / (1 - \mathbb{L}^a T^b)$ for $(a, b) \in \mathbb{Z} \times \mathbb{N}_{>0}$. An element of $\mathcal{M}[[T]]_{\Gamma}$ is called a *rational function*. Moreover, since

$$\frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b} = \frac{1}{\mathbb{L}^{-a} T^{-b} - 1},$$

we may expand a rational function in T^{-1} at 0 and take out the constant term of the resulting series to obtain an \mathcal{M} -linear morphism $\mathcal{M}[[T]]_{\Gamma} \rightarrow \mathcal{M}$, which is denoted

2010 *Mathematics Subject Classification.* 03C60, 14B20, 14E18, 14G22, 32S45.

Key words and phrases. formal schemes, definable subassignment, motivic integration, volume Poincaré series, motivic Milnor fiber, resolution of singularities.

as $\lim_{T \rightarrow \infty}$. This is the unique \mathcal{M} -linear morphism satisfying the property

$$\lim_{T \rightarrow \infty} \frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b} = -1$$

(cf. [2, Section 4]).

The motivic Milnor fiber of a formal function $(\mathfrak{X}, \mathfrak{f})$ at $x \in \mathfrak{X}_0$, where \mathfrak{X}_0 is the reduction of the formal scheme \mathfrak{X} , was defined by Kontsevich and Soibelman in [7] using numerical data of an embedded resolution of singularity of \mathfrak{f} at x , denoted as usual by $\mathcal{S}_{\mathfrak{f},x}$ (see [8] for more explanation). It is an element in the ring $\mathcal{M}_k^{\hat{\mu}}$. Moreover, Lê [8] extends this notion to motivic nearby cycles $\mathcal{S}_{\mathfrak{f}} \in \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}$ of a formal function $(\mathfrak{X}, \mathfrak{f})$. A result of Nicaise [13] on the rationality of volume Poincaré series for special formal schemes, after endowed with $\hat{\mu}$ -action by Lê [8], gives an alternative definition for $\mathcal{S}_{\mathfrak{f}}$ and $\mathcal{S}_{\mathfrak{f},x}$. In particular, it provides an effective description for $\mathcal{S}_{\mathfrak{f},x}$ in terms of the analytic Milnor fiber of \mathfrak{f} at x (introduced by Nicaise and Sebag [11]) via a group morphism MV between the Grothendieck ring of bounded smooth rigid varieties and $\mathcal{M}_k^{\hat{\mu}}$. More details on motivic Milnor fiber will be given in Section 2.1.

The purpose of this article is to prove the following result.

Theorem 1.1. *Let $f(x, y, z) \in k[[x, y, z]]$, where (x, y, z) is coordinates of the k -vector space $k^d = k^{d_1} \times k^{d_2} \times k^{d_3}$, invariant by the natural $\mathbb{G}_{m,k}$ -action of weight $(1, -1, 0)$ and $f(0, 0, 0) = 0$. Let \mathfrak{X} (resp. \mathfrak{Z}) be the formal completion of \mathbb{A}_k^d (resp. $\mathbb{A}_k^{d_3}$) along $\mathbb{A}_k^{d_1}$ (resp. at the origin) with structural morphism $f_{\mathfrak{X}}$ (resp. $f_{\mathfrak{Z}}$) induced by f (resp. $f(0, 0, z)$). Then the identity $\int_{\mathbb{A}_k^{d_1}} \mathcal{S}_{f_{\mathfrak{X}}} = \mathbb{L}^{d_1} \mathcal{S}_{f_{\mathfrak{Z}}}$ holds in $\mathcal{M}_{k,\text{loc}}^{\hat{\mu}}$.*

It is true that Theorem 1.1 is a crucial tool in constructing motivic Donaldson-Thomas invariants for noncommutative Calabi-Yau threefolds, widely known as the *integral identity conjecture* (cf. [7, Conjecture 4.4]). In fact, the statement of the theorem is weaker than that of the conjecture, since the setting of Theorem 1.1 is just $\mathcal{M}_{k,\text{loc}}^{\hat{\mu}}$, while the integral identity is expected to live in $\mathcal{M}_k^{\hat{\mu}}$. According to [7], Theorem 1.1 will be still enough good for the goal of use. This theorem was proved in [8] in the case where k is an algebraically closed field. At present our new method does not require that hypothesis of k .

Here is the idea of proof for Theorem 1.1. Using MV, as in [8, Section 7], the left hand side of the integral identity is nothing else than $\mathbb{L}^{d-1} \text{MV}([\mathfrak{X}_{\eta}])$, where \mathfrak{X}_{η} is the generic fiber of \mathfrak{X} , described as the set of elements (x, y, z) in the standard d -dimensional rigid space with valuation nonnegative for x , strictly positive for (y, z) and with $f(x, y, z) = t$ as well. Also by [8, Section 7], the rigid subvariety X_0 corresponding to $(x = 0 \text{ or } y = 0)$ gives rise to the right hand side of the integral identity with the formula $\mathbb{L}^{d_1} \mathcal{S}_{f_{\mathfrak{Z}}} = \mathbb{L}^{d-1} \text{MV}([X_0])$ (cf. also Theorem 3.1 in the loc. cit.).

The crucial method proposed in this article is to develop a result of Cluckers and Loeser on the rationality of a formal series (cf. [1, Theorem 4.4.1]). The work is based on the new foundation of motivic integration built up by Cluckers-Loeser [1] using model theory. In Section 4, we study a version with action for constructible motivic functions and state the rationality in that setting whose summary is placed in Corollary 4.7. Let X_1 be the complement of X_0 in \mathfrak{X}_{η} , which is also a rigid subvariety of \mathfrak{X}_{η} . By Theorem 3.1, proving Theorem 1.1 is now equivalent to proving Theorem 5.1, in Section 5, stating that $\text{MV}([X_1]) = 0$ in $\mathcal{M}_{k,\text{loc}}^{\hat{\mu}}$. We partition X_1

along the mapping $(x, y, z) \mapsto \text{val}(x) + \text{val}(y)$ into *summands*, in which the image of the class of each summand under MV vanishes (Lemma 5.2). By this, the proof then completes by applying Corollary 4.7.

2. MOTIVIC MILNOR FIBER AND VOLUME POINCARÉ SERIES

2.1. Motivic Milnor fiber. Let \mathfrak{X} be a generically smooth special formal $k[[t]]$ -scheme of relative dimension d , with reduction \mathfrak{X}_0 and structural morphism \mathfrak{f} (see [13] for definition). As explained in [8], this kind of formal scheme admits a resolution of singularities. So we may fix such a resolution \mathfrak{h} and consider the exceptional divisors, the irreducible components of the strict transform, all denoted as \mathfrak{E}_i 's, as well as their reductions $E_i = (\mathfrak{E}_i)_0$, where i is in a finite set J . For $\emptyset \neq I \subset J$, one puts

$$E_I = \bigcap_{i \in I} E_i \quad \text{and} \quad E_I^\circ = E_I \setminus \bigcup_{j \notin I} E_j.$$

Consider an affine covering $\{U\}$ of \mathfrak{h} such that on each piece $U \cap E_I^\circ \neq \emptyset$ the pullback of \mathfrak{f} has the form

$$u \prod_{i \in I} y_i^{N_i}$$

with u being a unit and y_i being a local coordinate defining E_i . The natural numbers N_i are nothing else than the multiplicity of the resolution \mathfrak{h} on \mathfrak{E}_i . Let m_I denote $\gcd(N_i)_{i \in I}$.

Definition 2.1. The Denef-Loeser unramified Galois covering $\pi_I : \tilde{E}_I^\circ \rightarrow E_I^\circ$ with Galois group μ_{m_I} is defined locally with respect to the covering $\{U\}$ as

$$\{(z, y) \in \mathbb{A}_k^1 \times (U \cap E_I^\circ) \mid z^{m_I} = u(y)^{-1}\}.$$

These local pieces are glued in an obvious way as in the proof of [2, Lemma 3.2.2] to build up \tilde{E}_I° and π_I .

Also due to the proof of [2, Lemma 3.2.2], the definition of the covering π_I is independent of the choice of the affine covering $\{U\}$ of \mathfrak{h} (of course, it depends on \mathfrak{h}). Moreover, each covering \tilde{E}_I° is endowed with a good μ_{m_I} -action by multiplication of the z -coordinate with elements of μ_{m_I} , thus defines a class $[\tilde{E}_I^\circ]$ in $\mathcal{M}_{\mathfrak{X}_0}^\mu$ (cf. [5]).

In [2], Denef and Loeser define the motivic nearby cycles and the motivic Milnor fiber of a regular function in terms of the corresponding motivic Igusa zeta function. Many classical invariants may be deduced from these notions, such as Euler characteristic, monodromy zeta function, Hodge spectrum. There is a fact that important applications mostly initiate from their description in the language of a resolution of singularities. Therefore, in the formal setting, together with the existence of resolution of singularities, this fact would be an inspiration for introducing the motivic Milnor fiber (in [7]) and the motivic nearby cycles (in [8]) of a formal function. More precisely, we recall these concepts from [7] and [8] in the following

Definition 2.2. The quantity

$$\sum_{\emptyset \neq I \subset J} (1 - \mathbb{L})^{|I|-1} [\tilde{E}_I^\circ]$$

in $\mathcal{M}_{\mathfrak{X}_0}^\mu$ is called the motivic nearby cycles of $(\mathfrak{X}, \mathfrak{f})$ and denoted by $\mathcal{S}_{\mathfrak{f}}$. If x is a closed point in \mathfrak{X}_0 , the motivic Milnor fiber $\mathcal{S}_{\mathfrak{f}, x}$ is defined as $x^* \mathcal{S}_{\mathfrak{f}}$ (in \mathcal{M}_k^μ), considering x as the inclusion of x in \mathfrak{X}_0 .

The below lemma shows that the motivic nearby cycles and the motivic Milnor fiber of a formal function are well defined.

Lemma 2.3 ([8], Lemma 5.7). *The previous definition of motivic nearby cycles and of motivic Milnor fiber is independent of the choice of resolution of singularities.*

2.2. Integrals and volume Poincaré series. Let \mathfrak{X} be a separated generically smooth formal $k[[t]]$ -scheme topologically of finite type, of pure relative dimension d . Motivic integral of a differential form on \mathfrak{X} was defined by Sebag [14] and developed in [9] and [11, 12]. To be useful for the proof of Theorem 1.1 let us go back to the original approach [14] (and [9]) and consider gauge forms.

By definition, in the category of formal $k[[t]]$ -schemes, \mathfrak{X} is the inductive limit of schemes of finite type

$$\mathcal{X}_m = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} \otimes_{k[[t]]} k[t]/(t^{m+1})), \quad m \in \mathbb{N}_{>0},$$

with transitions $\mathcal{X}_m \rightarrow \mathcal{X}_n$ being morphisms induced by the truncated morphisms $k[t]/(t^{n+1}) \rightarrow k[t]/(t^{m+1})$, for any $m \leq n$ in $\mathbb{N}_{>0}$. It was proved in [6] that the functor

$$\mathcal{Y} \mapsto \text{Hom}_{k[t]/(t^{m+1})}(\mathcal{Y} \times_k k[t]/(t^{m+1}), \mathcal{X}_m)$$

from the category of k -schemes to the category of sets is presented by a k -scheme $\text{Gr}_m(\mathcal{X}_m)$ topologically of finite type such that, for every k -algebra \mathcal{A} ,

$$\text{Gr}_m(\mathcal{X}_m)(\mathcal{A}) = \mathcal{X}_m(\mathcal{A} \otimes_k k[t]/(t^{m+1})).$$

These schemes together with obvious morphisms $\text{Gr}_n(\mathcal{X}_n) \rightarrow \text{Gr}_m(\mathcal{X}_m)$ for $m \leq n$ in $\mathbb{N}_{>0}$ establish a projective system, which we denote simply by $(\text{Gr}_m(\mathcal{X}_m))_{m \in \mathbb{N}}$. Let $\text{Gr}(\mathfrak{X})$ be the projective limit of $(\text{Gr}_m(\mathcal{X}_m))_{m \in \mathbb{N}}$ in the category of k -schemes, and let π_m be the canonical projection

$$\text{Gr}(\mathfrak{X}) \rightarrow \text{Gr}_m(\mathcal{X}_m),$$

for every $m \geq 1$.

Let ω be a gauge form on \mathfrak{X}_η . According to [9, Section 4.1], one can construct an integer-valued function $\text{ord}_{t,\mathfrak{X}}(\omega)$ on $\text{Gr}(\mathfrak{X})$ taking only a finite number of values with its fibers being *stable cylinders*, that is, for any $n \in \mathbb{Z}$, there exists an $l_0 \in \mathbb{N}_{>0}$ such that the quantity

$$D_n := [\pi_l(\text{ord}_{t,\mathfrak{X}}(\omega)^{-1}(n))] \mathbb{L}^{-(l+1)d} \in \mathcal{M}_{\mathfrak{X}_0}$$

does not depend on $l > l_0$.

Definition 2.4. *With \mathfrak{X} and ω as previous, one defines*

$$\int_{\mathfrak{X}} |\omega| := \sum_{n \in \mathbb{Z}} D_n \mathbb{L}^{-n} \in \mathcal{M}_{\mathfrak{X}_0}.$$

The following statement is referred to [9] and [11].

Proposition-Definition 2.5. *If X is a quasi-compact smooth rigid $k((t))$ -variety and ω is a gauge form on it, we may take a formal model \mathfrak{X} of X , which is a generically smooth formal $k[[t]]$ -scheme topologically of finite type, and define*

$$\int_X |\omega| := \int_{\mathfrak{X}_0} \left(\int_{\mathfrak{X}} |\omega| \right)$$

using the forgetful morphism $\int_{\mathfrak{X}_0} : \mathcal{M}_{\mathfrak{X}_0} \rightarrow \mathcal{M}_k$. It does not depend on the model \mathfrak{X} .

Now and later, let us assume that \mathfrak{X} is a generically smooth special formal $k[[t]]$ -scheme, of pure relative dimension d . According to [13], one may consider a Néron smoothening $\mathfrak{Y} \rightarrow \mathfrak{X}$ with \mathfrak{Y} being separated generically smooth topologically of finite type over $k[[t]]$. Using [13, Propositions 4.7, 4.8], for any gauge form ω on \mathfrak{X}_η , one defines

$$\int_{\mathfrak{X}} |\omega| := \int_{\mathfrak{Y}} |\omega|.$$

Definition 2.6. For $m \in \mathbb{N}_{>0}$, one defines

$$\mathfrak{X}(m) := \mathfrak{X} \widehat{\otimes}_{k[[t]]} k[[t^{1/m}]] \text{ and } \mathfrak{X}_\eta(m) := \mathfrak{X}_\eta \widehat{\otimes}_{k((t))} k((t^{1/m})).$$

One denotes by $\omega(m)$ the pullback of ω via the natural morphism $\mathfrak{X}_\eta(m) \rightarrow \mathfrak{X}_\eta$.

In fact, the Néron smoothening $\mathfrak{Y} \rightarrow \mathfrak{X}$ for \mathfrak{X} induces a Néron smoothening $\mathfrak{Y}(m) \rightarrow \mathfrak{X}(m)$ over $k[[t^{1/m}]]$, in which $\mathfrak{Y}(m)$ is also topologically of finite type, $m \in \mathbb{N}_{>0}$. As explained [8], the proof of Lemma 5.7, the scheme $\text{Gr}(\mathfrak{Y})(m)$ admits a good action of μ_m given by

$$\xi \varphi(t^{1/m}) := \varphi(\xi t^{1/m}).$$

This construction naturally takes $\int_{\mathfrak{X}(m)} |\omega(m)|$ to an element of $\mathcal{M}_{\mathfrak{X}_0}^{\mu_m}$. Assume that ω is an \mathfrak{X} -bounded gauge form on \mathfrak{X}_η , where the \mathfrak{X} -boundedness of ω was already defined in [13, Definition 2.11], which in present setting is just a technical tool for the computation ease and does not make any trouble for us. Now by means of the resolution of singularities mentioned in Section 2.1, let us put $\alpha_i := \text{ord}_{\mathfrak{E}_i} \omega$ for $i \in J$. The following result is the revisit of [13, Theorem 7.12] in the context with action.

Theorem 2.7. With the previous notation and hypotheses, it holds in $\mathcal{M}_{\mathfrak{X}_0}^{\mu_m}$ that

$$\int_{\mathfrak{X}(m)} |\omega(m)| = \mathbb{L}^{-d} \sum_{\emptyset \neq I \subset J} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^\circ] \left(\sum_{k_i \geq 1, \sum_{i \in I} k_i N_i = m} \mathbb{L}^{-\sum_{i \in I} k_i \alpha_i} \right).$$

Moreover, together with [13, Corollary 7.13], it implies

Theorem 2.8. With the previous notation and hypotheses, the series

$$\sum_{m \geq 1} \left(\int_{\mathfrak{X}(m)} |\omega(m)| \right) T^m \in \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}[[T]]$$

is a rational function, whose image under $\lim_{T \rightarrow \infty}$ is equal to $-\mathbb{L}^{-d} \mathcal{S}_f$ in $\mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}$.

In [12], Nicaise and Sebag introduce the notion of *bounded* rigid variety so that it is also applicable for the generic fiber of a special formal scheme. More precisely, a rigid $k((t))$ -variety X is bounded if there exists a quasi-compact open subspace Y of X such that $Y(K) = X(K)$ for any finite unramified extension K of $k((t))$. For such an X , and in addition smooth, we have that Y is also smooth. In this case, moreover, one can define the integral of a gauge form ω on X as follows (see [12])

$$\int_X |\omega| := \int_Y |\omega|,$$

where $\int_Y |\omega|$ was done in Proposition-Definition 2.5. In particular, for a special formal scheme \mathfrak{X} and a gauge form ω on \mathfrak{X}_η , we have

$$\int_{\mathfrak{X}_\eta} |\omega| = \int_{\mathfrak{X}_0} \left(\int_{\mathfrak{X}} |\omega| \right).$$

Let GBSRig be the category of gauged bounded smooth rigid $k((t))$ -varieties, as introduced in [8, Section 5.2]. Let BSRig be the category of bounded smooth rigid $k((t))$ -varieties, which may be obtained from GBSRig by forgetting gauge forms (cf. [8, Section 5.3]). The following theorem is a summary of the version with action of Loeser-Sebag's and Nicaise's results (cf. [9] and [13]).

Theorem 2.9 (Lê [8]).

(i) *The mapping*

$$\Phi : K(\text{GBSRig}) \rightarrow \mathcal{M}_k^{\hat{\mu}}$$

defined by $\Phi([X, \omega]) = \int_X |\omega|$ is a morphism of rings.

(ii) *If (X, ω) is an object in GBSRig with ω being \mathfrak{X} -bounded for some formal $k[[t]]$ -model \mathfrak{X} of X , then the series*

$$Z_{X, \omega}(T) := \sum_{m \geq 1} \Phi([X(m), \omega(m)]) T^m$$

in $\mathcal{M}_k^{\hat{\mu}}[[T]]$ is a rational function. The quantity $\lim_{T \rightarrow \infty} Z_{X, \omega}(T)$ is independent of the choice of ω .

(iii) *The mapping*

$$\text{MV} : K(\text{BSRig}) \rightarrow \mathcal{M}_k^{\hat{\mu}}$$

defined by $\text{MV}([X]) = -\lim_{T \rightarrow \infty} Z_{X, \omega}(T)$ is a morphism of groups. If $(\mathfrak{X}, \mathfrak{f})$ is a generically smooth special formal $k[[t]]$ -scheme of relative dimension d , then the identity

$$\text{MV}([\mathfrak{X}_\eta]) = \mathbb{L}^{-d} \int_{\mathfrak{X}_0} \mathfrak{S}_{\mathfrak{f}}$$

holds in $\mathcal{M}_k^{\hat{\mu}}$.

3. THE GEOMETRIC PART OF THE PROOF

In the present section, we shall use the notation and the hypotheses concerning Theorem 1.1 and recall some important points of computation in [8, Section 7]. By the homogeneity, the formal power series f belongs to the intersection ring $k\{x\}[[y, z]] \cap k\{y\}[[x, z]]$, and therefore, we may consider \mathfrak{X} as the formal completion of the $k[[t]]$ -formal scheme

$$\mathfrak{A} := \text{Spf}(k[[t]]\{x\}[[y, z]] \cap k[[t]]\{y\}[[x, z]]) / (f(x, y, z) - t)$$

along the subvariety $\mathbb{A}_k^{d_1}$ of the reduction \mathfrak{A}_0 . Thus \mathfrak{X} is a generically smooth special formal $k[[t]]$ -scheme of pure relative dimension $d - 1$, whose generic fiber \mathfrak{X}_η has the form

$$\mathfrak{X}_\eta = \left\{ (x, y, z) \in \mathbb{A}_{k((t))^{\text{alg}}}^{d, \text{an}} \mid \begin{array}{l} \text{val}(x) \geq 0, \text{val}(y) > 0, \text{val}(z) > 0 \\ f(x, y, z) = t \end{array} \right\},$$

where $\text{val}(x) := \min_{1 \leq i \leq d_1} \{\text{val}(x_i)\}$, the same for $\text{val}(y)$ and $\text{val}(z)$. Also, \mathfrak{Z} is the formal completion of the formal $k[[t]]$ -scheme

$$\text{Spfk}[[t, z]] / (f(0, 0, z) - t)$$

at the origin, whose generic fiber is the rigid variety

$$\{z \in \mathbb{A}_{k((t))}^{d_3, \text{an}} \mid \text{val}(z) > 0, f(0, 0, z) = t\}.$$

Let us now divide \mathfrak{X}_η into two disjoint definable subsets X_0 and X_1 subject to the conditions $(x = 0 \text{ or } y = 0)$ and $(x \neq 0 \text{ and } y \neq 0)$, respectively. The following theorem is the first step for proving the main theorem, Theorem 1.1.

Theorem 3.1. *With the previous hypotheses and notation, the identities*

$$\int_{\mathbb{A}_k^{d_1}} \mathcal{S}_{f_{\mathfrak{X}}} = \mathbb{L}^{d-1} \text{MV}([\mathfrak{X}_\eta]) \quad \text{and} \quad \mathbb{L}^{d_1} \mathcal{S}_{f_3} = \mathbb{L}^{d-1} \text{MV}([X_0])$$

hold in $\mathcal{M}_k^{\hat{\mu}}$. Therefore, Theorem 1.1 is equivalent to the identity $\text{MV}([X_1]) = 0$ in $\mathcal{M}_{k, \text{loc}}^{\hat{\mu}}$.

Proof. For a full demonstration of this theorem we refer to [8, Section 7], hence we only recall it here quickly as a sketch. The first identity is deduced directly from Theorem 2.9 (iii). To prove the second identity we look at the homogeneity of f , namely, if $x = 0$ or $y = 0$, $f(x, y, z) = f(0, 0, z)$. By this, we may write X_0 as the cartesian product $Y_0 \times \mathfrak{Z}_\eta$, where

$$Y_0 := \left(\{x \in \mathbb{A}_{k((t))}^{d_1, \text{an}} \mid \text{val}(x) \geq 0\} \setminus \{0\} \right) \sqcup \{y \in \mathbb{A}_{k((t))}^{d_2, \text{an}} \mid \text{val}(y) > 0\}.$$

In the previous decomposition, we denote the disjoint rigid subspaces of Y_0 by $Y_{0,1}$ and $Y_{0,2}$, respectively. Then we have

$$X_0 = (Y_{0,1} \times \mathfrak{Z}_\eta) \sqcup (Y_{0,2} \times \mathfrak{Z}_\eta).$$

Let dx denote the standard differential form $dx_1 \wedge \cdots \wedge dx_{d_1}$ on the closed ball $\{x \in \mathbb{A}_{k((t))}^{d_1, \text{an}} \mid \text{val}(x) \geq 0\}$, which is a gauge form. Now applying Theorem 2.7 and using the forgetful morphism we get the following identity in $\mathcal{M}_k^{\mu_m}$:

$$\Phi([Y_{0,1}(m), dx(m)]) = \Phi([\{x \in \mathbb{A}_{k((t^1/m))}^{d_1, \text{an}} \mid \text{val}(x) \geq 0\}, dx(m)]) - \Phi(1) = 0.$$

Let ω be any \mathfrak{Z} -bounded gauge form on \mathfrak{Z}_η . Then applying Theorem 2.9 and noting that Φ is a morphism of rings imply the vanishing of the below expression

$$\text{MV}([Y_{0,1} \times \mathfrak{Z}_\eta]) = - \lim_{T \rightarrow \infty} \sum_{m \geq 1} \Phi([Y_{0,1}(m) \times \mathfrak{Z}_\eta(m), dx(m) \times \omega(m)]) T^m.$$

Using the standard differential form $dy := dy_1 \wedge \cdots \wedge dy_{d_2}$ on the open ball $Y_{0,2}$ together with Theorem 2.7 we have

$$\Phi([Y_{0,2}(m), dy(m)]) = \mathbb{L}^{-d_2}$$

in $\mathcal{M}_k^{\mu_m}$. Again by Theorem 2.9,

$$\begin{aligned} \text{MV}([Y_{0,2} \times \mathfrak{Z}_\eta]) &= - \lim_{T \rightarrow \infty} \sum_{m \geq 1} \Phi([Y_{0,2}(m) \times \mathfrak{Z}_\eta(m), dy(m) \times \omega(m)]) T^m \\ &= -\mathbb{L}^{-d_2} \lim_{T \rightarrow \infty} \sum_{m \geq 1} \Phi([\mathfrak{Z}_\eta(m), \omega(m)]) T^m = \mathbb{L}^{-d_2-d_3+1} \mathcal{S}_{f_3}. \end{aligned}$$

Hence $\mathbb{L}^{d-1} \text{MV}([X_0]) = \mathbb{L}^{d_1} \mathcal{S}_{f_3}$ in $\mathcal{M}_k^{\hat{\mu}}$. □

4. A THEOREM OF CLUCKERS-LOESER

4.1. The setting of constructible motivic functions. In the present paragraph, let us recall definitions, notation and results in [1] which are useful for our goal. Definition 4.3 is to promote a new notion.

The formalism of Cluckers and Loeser is based on the Denef-Pas language \mathcal{L}_{DP} with the ring language for valued fields and residue fields and with the Presburger language for value groups (cf. [1]). Let \mathbf{T} be a theory generated by sentences in \mathcal{L}_{DP} with coefficients in k and $k((t))$. Denote by $\text{Field}_k(\mathbf{T})$ the category of fields K containing k so that $(K((t)), K, \mathbb{Z})$ is a model of \mathbf{T} , its morphisms are field morphisms. Such models are endowed with order and angular component mappings. A primary definable \mathbf{T} -subassignment has the form

$$h[m, n, r](K) := K((t))^m \times K^n \times \mathbb{Z}^r.$$

For a formula φ in \mathcal{L}_{DP} , the subset of $h[m, n, r](K)$ defined by φ is denoted by $h_\varphi(K)$. More generally, if $W = \mathcal{X} \times \mathcal{X} \times \mathbb{Z}^r$ with \mathcal{X} a $k((t))$ -variety, \mathcal{X} a k -variety, one writes

$$h_W(K) := \mathcal{X}(K((t))) \times \mathcal{X}(K) \times \mathbb{Z}^r.$$

We shall omit \mathbf{T} if no confusion may arise.

Definition 4.1. *The category of all the definable \mathbf{T} -subassignments $K \mapsto h_\varphi(K)$ is denoted by Def_k or more precisely by $\text{Def}_k(\mathcal{L}_{\text{DP}}, \mathbf{T})$. For any object S in Def_k , let Def_S or $\text{Def}_S(\mathcal{L}_{\text{DP}}, \mathbf{T})$ be the category of objects of Def_k over S and let RDef_S or $\text{RDef}_S(\mathcal{L}_{\text{DP}}, \mathbf{T})$ be the subcategory of Def_S whose objects are subassignments of $S \times h_{\mathbb{A}_k^n}$, for variable n , morphisms to S are the ones induced by the projection on the S -factor.*

Definition 4.2. *The Grothendieck group $K_0(\text{RDef}_S)$ is a free abelian generated by symbols $[X \rightarrow S]$ with $X \rightarrow S$ in RDef_S modulo the conditions:*

$$[X \rightarrow S] = [Y \rightarrow S]$$

if $[X \rightarrow S]$ and $[Y \rightarrow S]$ are isomorphic in RDef_S ,

$$[X \cup Y \rightarrow S] + [X \cap Y \rightarrow S] = [X \rightarrow S] + [Y \rightarrow S]$$

for any definable \mathbf{T} -subassignments X and Y of $S \times h_{\mathbb{A}_k^n}$ for some $n \in \mathbb{N}$.

Let $X \rightarrow S$ be an object in RDef_S and let $m \in \mathbb{N}_{>0}$. Assume $X = h_W$ with $W = \mathcal{X} \times \mathcal{X} \times \mathbb{Z}^r$ for some $k((t))$ -variety \mathcal{X} , k -variety \mathcal{X} and some $r \in \mathbb{N}$. For an algebraic group scheme \mathcal{G} , a \mathcal{G} -action on X is a morphism

$$h_{\mathcal{G}} \times X \rightarrow X$$

of subassignments over S . A *good μ_m -action* on X is a μ_m -action on X such that each orbit intersected with $h_{\mathcal{X}}$ is contained in $h_{\mathcal{V}}$ with \mathcal{V} an affine subvariety of \mathcal{X} . A *good $\hat{\mu}$ -action* on X is a $\hat{\mu}$ -action on X that factors through a good μ_m -action on X , for some $m \in \mathbb{N}_{>0}$.

Definition 4.3. *The (monodromic) Grothendieck group $K_0^{\hat{\mu}}(\text{RDef}_S)$ is a free abelian generated by symbols $[X \rightarrow S, \hat{\mu}]$ (or $[X, \hat{\mu}]$ or $[X]$ for short) with $(X \rightarrow S)$ in RDef_S , X endowed with a good $\hat{\mu}$ -action, with the relations as in Definition 4.2, augmented one more condition:*

$$[X \times h_{\mathcal{V}}, \hat{\mu}] = [X \times h_{\mathbb{A}_k^n}, \hat{\mu}],$$

where \mathcal{V} is the n -dimensional affine k -space endowed with an arbitrary linear $\hat{\mu}$ -action and \mathbb{A}_k^n with trivial $\hat{\mu}$ -action, for any $n \in \mathbb{N}$. The groups $K_0(\text{RDef}_S)$ and $K_0^{\hat{\mu}}(\text{RDef}_S)$ become rings with respect to fiber product of subassignments (cf. [1, Section 2.2]).

Put

$$\mathbf{A} := \mathbb{Z} [\mathbb{L}, \mathbb{L}^{-1}, (1 - \mathbb{L}^{-i})^{-1}, i > 0],$$

in which at the moment we consider \mathbb{L} as a formal symbol. For S in Def_k , let $\mathcal{P}(S)$ be the subring of the ring of functions $S \rightarrow \mathbf{A}$ generated by all constant functions into \mathbf{A} , all definable functions $S \rightarrow \mathbb{Z}$ and all functions of the form \mathbb{L}^α with α being a \mathbb{Z} -valued definable function on S (each element of $\mathcal{P}(S)$ is called a *Presburger function*). Note that, if S is one point, one may identify $\mathcal{P}(S)$ with \mathbf{A} . Let $\mathcal{P}^0(S)$ be the subring of $\mathcal{P}(S)$ generated by $\mathbb{L} - 1$ and by character functions $\mathbf{1}_Y$ for all definable subassignments Y of S .

Definition 4.4. The ring of constructible motivic functions on S and its monodromic version are defined as follows

$$\mathcal{C}(S) := K_0(\text{RDef}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S) \text{ and } \mathcal{C}^{\hat{\mu}}(S) := K_0^{\hat{\mu}}(\text{RDef}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S).$$

Let us consider the case where $\mathbf{T} = \mathbf{T}_{\text{acl}}$, the theory of algebraically closed fields containing k , as in [1, Sections 16.2, 16.3]. We also rewrite $\text{RDef}_{h_X}(\mathcal{L}_{\text{DP}}, \mathbf{T}_{\text{acl}})$ simply by RDef_{h_X} .

Proposition 4.5. For any algebraic k -variety \mathcal{X} , there are canonical isomorphisms of rings $K_0(\text{RDef}_{h_X}) \cong K_0(\text{Var}_{\mathcal{X}})$, $\mathcal{C}(h_X) \cong \mathcal{M}_{\mathcal{X}, \text{loc}}$, $K_0^{\hat{\mu}}(\text{RDef}_{h_X}) \cong K_0^{\hat{\mu}}(\text{Var}_{\mathcal{X}})$ and $\mathcal{C}^{\hat{\mu}}(h_X) \cong \mathcal{M}_{\mathcal{X}, \text{loc}}^{\hat{\mu}}$.

Proof. The first two isomorphisms are shown in [1, Sections 16.2, 16.3] due to definition. For the first case with action, let Y be an arbitrary subassignment of $h_X \times h_{\mathbb{A}_k^n} = h_{X \times \mathbb{A}_k^n}$, $n \in \mathbb{N}$. There exists a unique subvariety \mathcal{Y} of $X \times \mathbb{A}_k^n$ such that $Y = h_{\mathcal{Y}}$. If $h_{\hat{\mu}} \times Y \rightarrow Y$ is a good $\hat{\mu}$ -action on Y , there exists a unique $\hat{\mu}$ -action $\hat{\mu} \times \mathcal{Y} \rightarrow \mathcal{Y}$ on \mathcal{Y} compatible with it via the canonical isomorphism $K_0(\text{RDef}_{h_X}) \cong K_0(\text{Var}_{\mathcal{X}})$, hence we have

$$K_0^{\hat{\mu}}(\text{RDef}_{h_X}) \cong K_0^{\hat{\mu}}(\text{Var}_{\mathcal{X}}).$$

The fourth isomorphism is a consequence of the third one. \square

4.2. Rationality of formal series. In what follows we refer to [1, Sections 4.4–5.7] for a result on the rationality of a formal series and state the version with action of this result. Let S be an object in Def_k , let $r \in \mathbb{N}_{>0}$ and let $T = (T_1, \dots, T_r)$ be variables. Consider the ring $\mathcal{C}(S)[[T]]$ of formal series with coefficients in $\mathcal{C}(S)$. For a definable function $\alpha : S \rightarrow \mathbb{N}^r$, let

$$T^\alpha := \sum_{j \in \mathbb{N}^r} \mathbf{1}_{C_j} T^j,$$

where

$$C_j := \{x \in S \mid \alpha(x) = j\}.$$

Denote by $\mathcal{C}(S)\{T\}$ the $\mathcal{C}(S)$ -subalgebra of $\mathcal{C}(S)[[T]]$ generated by the series T^α with $\alpha : S \rightarrow \mathbb{N}^r$ definable. Let Γ be the multiplicative set of polynomials in $\mathcal{C}(S)[T]$ generated by $1 - \mathbb{L}^a T^b$ with $(a, b) \in \mathbb{Z} \times \mathbb{N}^r$, $b \neq 0$. We denote by $\mathcal{C}(S)\{T\}_\Gamma$

the localization of $\mathcal{C}(S)\{T\}$ with respect to Γ and by $\mathcal{C}(S)[[T]]_\Gamma$ the image of the injective morphism of rings

$$\mathcal{C}(S)\{T\}_\Gamma \rightarrow \mathcal{C}(S)[[T]].$$

Thus, one may identify $\mathcal{C}(S)[[T]]_\Gamma$ with $\mathcal{C}(S)\{T\}_\Gamma$.

One also considers the $\mathcal{C}(S)$ -module $\mathcal{C}(S)[[T, T^{-1}]]$. It has a ring structure by the Hadamard product: if $f = \sum_{i \in \mathbb{Z}^r} a_i T^i$ and $g = \sum_{i \in \mathbb{Z}^r} b_i T^i$ are in $\mathcal{C}(S)[[T, T^{-1}]]$, one defines

$$f * g := \sum_{i \in \mathbb{Z}^r} a_i b_i T^i.$$

The subrings $\mathcal{C}(S)[[T]]_\Gamma$ and $\mathcal{C}(S)[[T, T^{-1}]]_\Gamma$ are stable by the Hadamard product (deduced from [1, Remark 4.4.2]).

For $\varphi \in \mathcal{C}(S \times \mathbb{Z}^r)$ and $i \in \mathbb{Z}^r$, or for $\varphi \in \mathcal{C}(S \times \mathbb{N}^r)$ and $i \in \mathbb{N}^r$, we denote by φ_i the restriction of φ to $S \times \{i\}$ and consider it as an element of $\mathcal{C}(S)$. We define

$$M(\varphi) := \sum_{i \in \Delta} \varphi_i T^i,$$

which is a series in $\mathcal{C}(S)[[T, T^{-1}]]$, where $\Delta = \mathbb{Z}^r$ or $\Delta = \mathbb{N}^r$, depending on contexts.

Theorem 4.6. *Let S be an object in Def_k . The correspondences $\varphi \mapsto M(\varphi)$ induce isomorphisms of rings*

$$\mathcal{C}^\mu(S \times \mathbb{Z}^r) \rightarrow \mathcal{C}^\mu(S)[[T, T^{-1}]]_\Gamma \text{ and } \mathcal{C}^\mu(S \times \mathbb{N}^r) \rightarrow \mathcal{C}^\mu(S)[[T]]_\Gamma.$$

Proof. These isomorphisms without action are deduced from [1, Theorem 4.4.1], with $\mathcal{P}(S)$ being the coefficient ring, by noting that

$$\mathcal{C}(S) = K_0(\text{RDef}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S)$$

(cf. also [1, Theorem 5.7.1]). Remark that, by definition, the action structure in $\mathcal{C}^\mu(S)$ is induced by the one in $K_0^\mu(\text{RDef}_S)$, hence we may still use the previous arguments for the case with action. \square

Let us now consider the theory \mathbf{T}_{acl} of algebraically closed fields containing k , as in [1, Sections 16.2, 16.3]. As above let \mathcal{X} be an algebraic k -variety. Combining Proposition 4.5 and Theorem 4.6 we have the following corollary.

Corollary 4.7. *The mapping $\mathcal{C}^\mu(h_{\mathcal{X}} \times \mathbb{N}^r) \rightarrow \mathcal{M}_{\mathcal{X}, \text{loc}}^\mu[[T]]_\Gamma$ defined by $\varphi \mapsto M(\varphi)$ is an isomorphism of rings.*

5. PROOF OF THEOREM 1.1

In this section we prove the following theorem, which together with Theorem 3.1 then implies Theorem 1.1.

Theorem 5.1. *With the notation in Section 3, $\text{MV}([X_1]) = 0$ in $\mathcal{M}_{k, \text{loc}}^\mu$.*

Proof. First, we claim that $\text{MV}([X_1])$ may be viewed as an element of $\mathcal{C}^\mu(\mathbb{N}_{>0})$ with structural mapping θ induced by $(x, y, z) \mapsto \text{val}(x) + \text{val}(y)$. Indeed, Theorem 2.9 gives the following expression

$$\text{MV}([X_1]) = - \lim_{T \rightarrow \infty} \sum_{m \geq 1} \Phi([X_1(m), \omega(m)]) T^m$$

with ω being an appropriate gauge form on X_1 . Let \mathfrak{X}' be a formal model of X_1 . By using a Néron smoothening for it according to Proposition-Definition 2.5, we may

assume that \mathfrak{X}' is a generically smooth formal scheme topologically of finite type over $k[[t]]$. By Definition 2.4 and Proposition-Definition 2.5, for ℓ large enough, $\Phi([X_1(m), \omega(m)])$ is equal to

$$\int_{\mathfrak{X}'_0} \sum_{n \in \mathbb{Z}} [\{(x, y, z) \in \text{Gr}_\ell(\mathfrak{X}'(m)) \mid \text{ord}_{t^{1/m}, \mathfrak{X}'(m)}(\omega(m))(x, y, z) = n\} \rightarrow \mathfrak{X}'_0].$$

Thus the correspondence

$$(x, y, z) \mapsto \text{ord}_{t^{1/m}}(x) + \text{ord}_{t^{1/m}}(y)$$

defines a mapping

$$\theta_m : \Phi([X_1(m), \omega(m)]) \rightarrow \mathbb{N}_{>0}$$

for each $m \in \mathbb{N}_{>0}$, and all of these mappings have the same value at all the m -ramifications of a point in $\text{MV}([X_1])$. Here, for abuse of notation, we identify $\Phi([X_1(m), \omega(m)])$ and $\text{MV}([X_1])$ with their representative, respectively. Thus all the mappings θ_m define a mapping

$$\theta : \text{MV}([X_1]) \rightarrow \mathbb{N}_{>0},$$

which takes $\text{MV}([X_1])$ to an element of $\mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0})$ with structure as desired. The claim is proved.

For any $n \in \mathbb{N}_{>0}$, $\theta_m^{-1}(n)$ is a definable subset of $\Phi([X_1(m), \omega(m)])$ defined by $\text{val}(x) + \text{val}(y) = n$. Thus $\theta^{-1}(n) = \text{MV}([X_{1,n}])$ with

$$X_{1,n} := \bigcup_{m \geq 1} \{(x, y, z) \in X_1 \mid \text{val}(x) + \text{val}(y) = \frac{n}{m}\}.$$

Lemma 5.2. *For all $n, m \in \mathbb{N}_{>0}$,*

$$\text{MV}(\{(x, y, z) \in X_1 \mid \text{val}(x) + \text{val}(y) = \frac{n}{m}\}) = 0$$

in $\mathcal{M}_k^{\hat{\mu}}$.

Proof of Lemma 5.2. Denote by Y the set $\{(x, y, z) \in X_1 \mid \text{val}(x) + \text{val}(y) = \frac{n}{m}\}$. Consider the action of $G := \mathbb{G}_{m, k((t))}^{\text{an}}_{\text{alg}}$ on the rigid variety

$$Z := \left(\mathbb{A}_{k((t))_{\text{alg}}}^{d_1, \text{an}} \setminus \{0\} \right) \times \left(\mathbb{A}_{k((t))_{\text{alg}}}^{d_2, \text{an}} \setminus \{0\} \right) \times \mathbb{A}_{k((t))_{\text{alg}}}^{d_3, \text{an}}$$

defined by

$$\tau \cdot (x, y, z) := (\tau x, \tau^{-1} y, z)$$

for $\tau \in G$. It is clear that this is a free action. Let \tilde{Y} be the image of Y under the projection $Z \rightarrow Z/G$. Then the induced mapping $Y \rightarrow \tilde{Y}$ is a fibration with fiber

$$\{(\tau x, \tau^{-1} y, z) \mid -\text{val}(x) \leq \text{val}(\tau) < \text{val}(y)\}$$

over the class of (x, y, z) , and the fiber is isomorphic to

$$A_{\frac{n}{m}} := \left\{ \tau \in G \mid 0 \leq \text{val}(\tau) < \frac{n}{m} \right\}$$

since, in Y , $\text{val}(x) + \text{val}(y) = \frac{n}{m}$, and since all the annuli of this kind having the same modulus are isomorphic. Note that $[Y] = [\tilde{Y}][A_{\frac{n}{m}}] = [\tilde{Y} \times A_{\frac{n}{m}}]$ and that Φ is

a morphism of rings (while MV is just a morphism of groups), we have

$$\begin{aligned} MV([Y]) &= - \lim_{T \rightarrow \infty} \sum_{l \geq 1} \Phi([Y(l), \omega(l)]) T^l \\ &= - \lim_{T \rightarrow \infty} \sum_{l \geq 1} \Phi([\tilde{Y}(l), \tilde{\omega}(l)]) \Phi([A_{\frac{n}{m}}(l), d\tau(l)]) T^l \end{aligned}$$

for some gauge forms ω on Y , $\tilde{\omega}$ on \tilde{Y} . Thus, by [10, lemma 7.6], we have

$$MV([Y]) = \left(\lim_{T \rightarrow \infty} \sum_{l \geq 1} \Phi([\tilde{Y}(l), \tilde{\omega}(l)]) T^l \right) \cdot \left(\lim_{T \rightarrow \infty} \sum_{l \geq 1} \Phi([A_{\frac{n}{m}}(l), d\tau(l)]) T^l \right).$$

The second factor vanishes in $\mathcal{M}_k^{\hat{\mu}}$ in the following reason. Note that the annulus $A_{\frac{n}{m}}$ is the difference of the rigid varieties

$$B := \{\tau \in \mathbb{A}_{k((t))}^{1, \text{an}} \mid \text{val}(\tau) \geq 0\} \text{ and } B' := \{\tau \in \mathbb{A}_{k((t))}^{1, \text{an}} \mid \text{val}(\tau) \geq \frac{n}{m}\},$$

in which $\Phi([B(l), d\tau(l)]) = 1$ (applying directly Theorem 2.7). Assume n and m are coprime and remark that, if m does not divide l , $\Phi([B'(l), d\tau(l)]) = 0$. Now assume $l = me$ with $e \geq 1$. Then, by using Theorem 2.7, one may show that $\Phi([B'(me), d\tau(me)]) = \mathbb{L}^{-ne}$. Thus

$$\sum_{l \geq 1} \Phi([A_{\frac{n}{m}}(l), d\tau(l)]) T^l = \sum_{l \geq 1} T^l - \sum_{e \geq 1} \mathbb{L}^{-ne} T^{me},$$

whose image under $\lim_{T \rightarrow \infty}$ vanishes, implying the vanishing of $MV([Y])$ in $\mathcal{M}_k^{\hat{\mu}}$. \square

Lemma 5.3. *With the previous notation, for any $n \in \mathbb{N}_{>0}$, $\theta^{-1}(n) = 0$ as an element of $\mathcal{M}_{k, \text{loc}}^{\hat{\mu}}$.*

Proof of Lemma 5.3. Consider the mapping

$$s_n : X_{1,n} \rightarrow \mathbb{N}_{>0}$$

defined uniquely by the condition: if $\text{val}(x) + \text{val}(y) = \frac{n}{m}$ then $s_n(x, y, z) = m$. Then using the same arguments as in the proof of Lemma 5.2, one may consider the quantity $\theta^{-1}(n) = MV([X_{1,n}])$ as an element of $\mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0})$ admitting structural mapping

$$\lambda_n : \theta^{-1}(n) \rightarrow \mathbb{N}_{>0}$$

induced by s_n . For any $m \in \mathbb{N}_{>0}$, we have

$$\lambda_n^{-1}(m) = MV(\{(x, y, z) \in X_1 \mid \text{val}(x) + \text{val}(y) = \frac{n}{m}\}) = 0,$$

thanks to Lemma 5.2. By applying Corollary 4.7, we obtain an isomorphism of rings

$$M : \mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0}) \rightarrow \mathcal{M}_{k, \text{loc}}^{\hat{\mu}}[[T]]_{\Gamma}$$

and an identity

$$M(\theta^{-1}(n)) = \sum_{m \geq 1} \lambda_n^{-1}(m) T^m = 0,$$

which implies that $\theta^{-1}(n) = 0$ in $\mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0})$. Finally, passing through the forgetful morphism, we get $\theta^{-1}(n) = 0$ in $\mathcal{M}_{k, \text{loc}}^{\hat{\mu}}$. \square

We apply Corollary 4.7 with the isomorphism

$$M : \mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0}) \rightarrow \mathcal{M}_{k,\text{loc}}^{\hat{\mu}}[[T]]_{\Gamma}$$

to $\text{MV}([X_1])$. Together with Lemma 5.3, we get

$$M(\text{MV}([X_1])) = \sum_{m \geq 1} \theta^{-1}(n) T^n = 0,$$

hence $\text{MV}([X_1]) = 0$ in $\mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0})$. Passing through the forgetful morphism, it implies that $\text{MV}([X_1]) = 0$ in $\mathcal{M}_{k,\text{loc}}^{\hat{\mu}}$. Theorem 5.1 is now completely proved. \square

Acknowledgements. Part of this article was done at the Max Planck Institute for Mathematics in Bonn, the author would like to thank the institute for warm hospitality. He thanks François Loeser for communications and encouragement.

REFERENCES

- [1] R. Cluckers and F. Loeser, *Constructible motivic functions and motivic integration*, Invent. Math. **173** (2008), no. 1, 23–121.
- [2] J. Denef and F. Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), 505–537.
- [3] J. Denef and F. Loeser, *Germes of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), 201–232.
- [4] J. Denef and F. Loeser, “*Geometry on arc spaces of algebraic varieties*” in European congress of Mathematics, Vol. 1 (Barcelona, 2000), Progr. Math. **201**, Birkhäuser, Basel, 2001, 327–348.
- [5] J. Denef and F. Loeser, *Lefschetz numbers of iterates of the monodromy and truncated arcs*, Topology **41** (2002), no. 5, 1031–1040.
- [6] M. J. Greenberg, *Schemata over local rings*, Ann. of Math. **73** (1961), 624–648.
- [7] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv: 0811.2435v1.
- [8] Q. T. Lê, *Proofs of the integral identity conjecture over algebraically closed fields*, Duke Math. J., **164** No. 1 (2015), 157–194.
- [9] F. Loeser and J. Sebag, *Motivic integration on smooth rigid varieties and invariants of degenerations*, Duke Math. J. **119** (2003), no. 2, 315–344.
- [10] E. Looijenga, *Motivic measures*, Astérisque **276** (2002), 267–297, Séminaire Bourbaki 1999/2000, no. 874.
- [11] J. Nicaise and J. Sebag, *Motivic Serre invariants, ramification, and the analytic Milnor fiber*, Invent. Math. **168** (2007), no. 1, 133–173.
- [12] J. Nicaise and J. Sebag, *Motivic Serre invariants and Weil restriction*, J. Algebra. **319** (2008), 1585–1610.
- [13] J. Nicaise, *A trace formula for rigid varieties, and motivic Weil generating series for formal schemes*, Math. Ann. **343** (2009), 285–349.
- [14] J. Sebag, *Intégration motivique sur les schémas formels*, Bull. Soc. Math. France **132** (2004), no. 1, 1–54, Séminaire Bourbaki 1999/2000, no. 874.

DEPARTMENT OF MATHEMATICS, VIETNAM NATIONAL UNIVERSITY, 334 NGUYEN TRAI STREET, HANOI, VIETNAM

E-mail address: leqthuong@gmail.com